Lecture Notes (Math 90): Week I

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1 Introduction and general nonsense

MATH 90 is the first part of an introductory calculus for individuals with no previous experience in the subject. Some pre-calculus background is generally assumed, including trigonometric functions and hopefully an acquaintance with exponentials and logarithms (Though we shall briefly review these topics below). In this semester we will cover limits, differentiation, and the beginning of integrations. The second semester covers more advanced integration techniques together with sequences and series

Most of the general information about MATH 90 may be found on the handout stapled to the back.

1.1 Contact information

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My lecture notes will eventually appear on my personal website.

1.2 Office hours

Office hours will be held for two hours each week at a fixed time, though I am often available to meet outside the appointed times if you cannot make it on a given way (Shoot me an email). The times for office hours are *to be decided*. Please vote in the following poll for the time that works best for you.

Office hours poll: *http://doodle.com/poll/uzfvu96m6p6xi577*

2 Interlude: Philosophical Overview

If you have absolutely no experience with calculus, the first and most natural question to pose - the very question you may be thinking at this moment - is simply: Why should *I* care?

To answer this, one might start by imagining a vast network of vessels and flows between them. Everything in flux! As a single vessels empties, another elsewhere is filled. One example is given by network of all the waterways, stream, and tributaries in Brazil together with the larger bodies of water which they serve to connect. Another is the flow of capital between territories and nations. Yet another still is the movement of electrical signals across a circuit board.

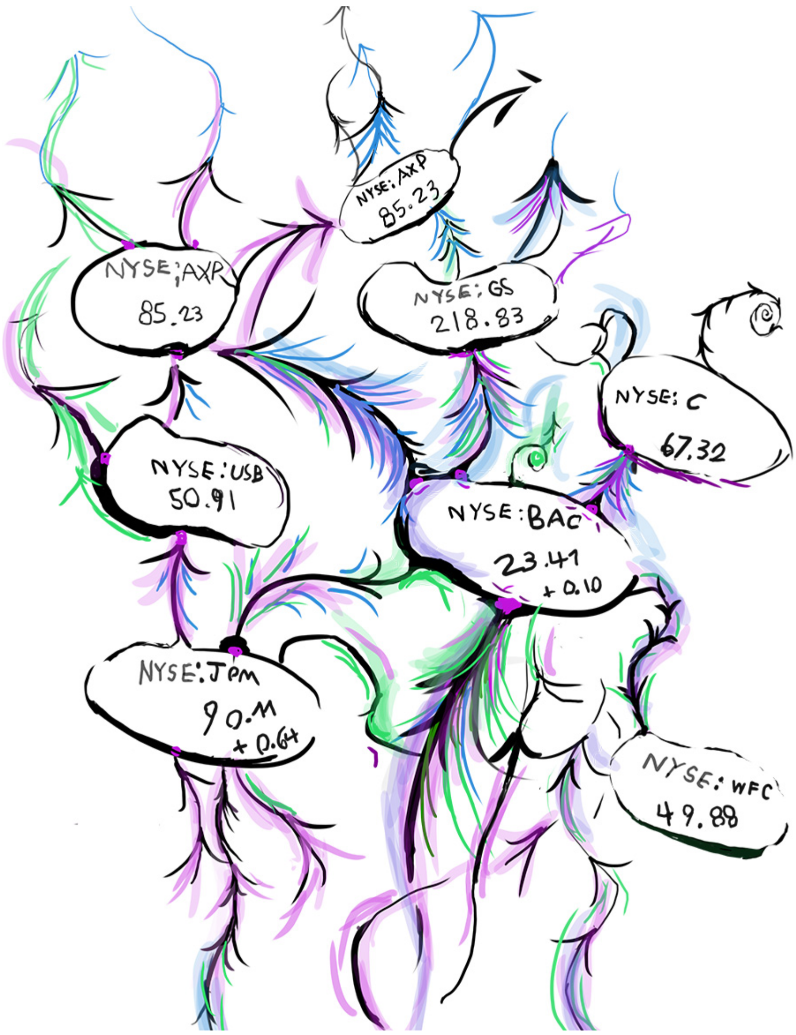

Figure 1: An abstract network. The closing prices of major banks on September 7th, 2017

The only general method to approach such a complex system amounts to looking at it microscopically and focusing on a single vessel or a pair of vessels and ask about the magnitude of the quantity inside each. Or perhaps we may ask if there is a relationship between the magnitude of one with the magnitude of the other, is there a quantity *flowing* from *vessel A* to *vessel B*? The first question concerns *accumulation* and the second concerns *change*.

Both questions implicitly concern time. The first question concerns the quantity accumulated in *vessel A* and in *vessel B* at a time t. The second question concerns the rate of flow of the quantity from *vessel A* to *vessel B* at a time t. In a sense, calculus is built around the following observation: If you know the answer to the first question *at all times t*, you may calculate the answer to the second question *at all times t*. Moreover, if you know the answer to the second question *at all times* t together with the quantities in the vessels at a specified time t_0, then you may calculate the answer to the first question!

Here is an easy example of how the above concepts work in practice: Imagine a river connecting two distinct lakes. Let's call them lake Alpha and lake Omega. Water flows down from Alpha to Omega at a constant rate. If we know how much water is in each of the lakes at each point in time, can you say what the *flow rate* is? Let $Vol(Alpha, t)$ be the cubic gallons of water in lake Alpha at time t and suppose that $t_0 < t_1$. Then, since the rate of change is given by $\frac{difference}{time}$, we see that lake Alpha is draining at a rate

$$s = \frac{Vol(Alpha, t_1) - Vol(Alpha, t_0)}{t_1 - t_0}$$

On the other hand, if you know that $Vol(Alpha, 0) = 1000$ and you know that lake Alpha is draining at a rate s, then amount of water in lake Alpha at time t is $100 - st$. So, we have thus computed that

$$Vol(Alpha, t) = 1000 - st$$

The above example is not particularly difficult, but it captures the fundamental ideas and theorems of calculus. The only difference is that honest calculus no longer assumes the rate of change to be constant, and that means we have to work much harder to make sense of the relation between *change* and *accumulation*.

3 Special functions

3.1 Trigonometric Functions

We will not review trigometry very much except to consider the following special values for which $sin(x)$ and $cos(x)$ may be easily computed (see the figure below) together with some of the important relations between them.

First off, the most important relation between sine and cosine is given by the following formula

$$sin(x)^2 + cos(x)^2 = 1$$

This relationship tells us that $(sin(x), cos(x))$ parametrizes points on the unit circle. Think of them as machines that take an angle $x$as input, and together output a point on the unit circle.

The second relation we shall consider are the double angle formulas and they tell us that

$$cos(2x) = cos(x)^2 - sin(x)^2$$

$$sin(2x) = 2cos(x)sin(x)$$

These relations, together with the special values on the unit circle, make it possible to compute sin and cosine for a wide variety of special angles.

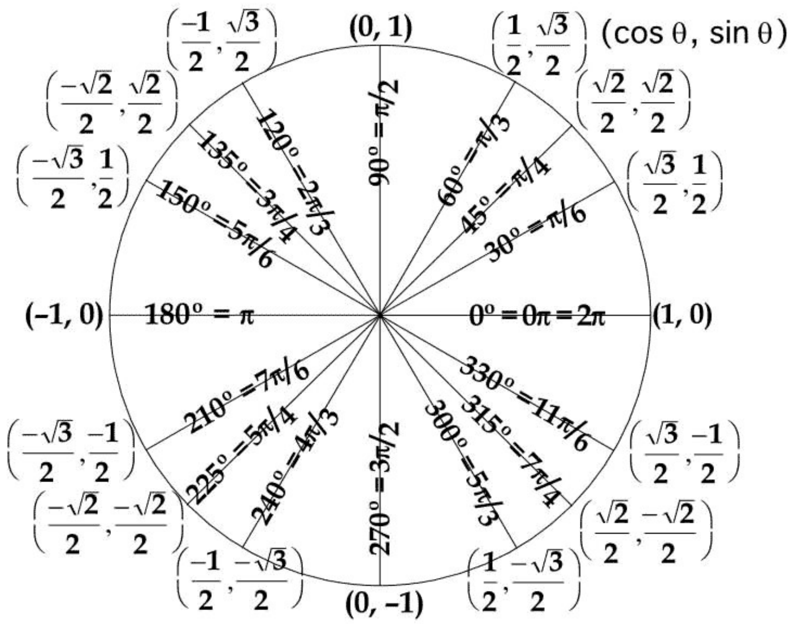

Figure 2: Special values of trigonometric functions

3.2 Exponentials and logarithms

The exponential function $f(x) = e^x$ has often been called *the most important function in mathematics*. Before we consider it, let us recall the simple act of raising a number n to a power b. We all know that $2^2 = 4$ and that $2^3 = 8$, perhaps some also know that $2^{31} = 2147483648$. Thus, for every natural number b, I obtain a number 2^b given by multiplying 2 by itself b times. But what about, say, $2^{\frac{1}{2}}$? How may we think of a number raised to a fraction?

The answer involves another familiar operation, square roots! So, we shall regard $2^{\frac{1}{2}}$ as the same thing as $\sqrt{2}$. Likewise, we shall regard $2^{\frac{1}{3}}$ as the cubic root of 2. Thus, we might regard $2^{\frac{a}{b}}$ as the number obtained by first taking the bth root of 2, and then raising it to the ath power.

Example. Suppose we wish to write down a decimal approximation of $2^{\frac{3}{2}}$ using a calculator. First we compute $2^{\frac{1}{2}} = \sqrt{2} = 1.41421\ldots$. Then we raise 1.41421 to the third power:

$$(1.41421\ldots)(1.41421\ldots)(1.41421\ldots) = 2.82842\ldots$$

Thus, 2.8284 is a decimal approximation of the number $2^{\frac{3}{2}}$. In practice, most calculators would simply allow you to compute $2^{\frac{3}{2}}$ immediately.

We also take $n^0 = 1$ by tradition and that $n^{\frac{a}{b}} = (\frac{1}{n})^{-\frac{a}{b}}$ when the rational number $\frac{a}{b}$ is negative. Thus, we have a method for computing $n^{\frac{a}{b}}$ for any rational number $\frac{a}{b}$, not just the positive ones!

With the case of rational numbers settled, what about raising a number to an *irrational* number? For example, what could one possibly mean by 2^π? Well, this involves the notion of *limits*, a central topic of this course. For now, you might think of 2^π in the following practical way. Find a bunch of fractions $\frac{a_n}{b_n}$ approximating π, then $2^{\frac{a}{b}}$ also approximates 2^π. If this is confusing, then it may help to study the following graph

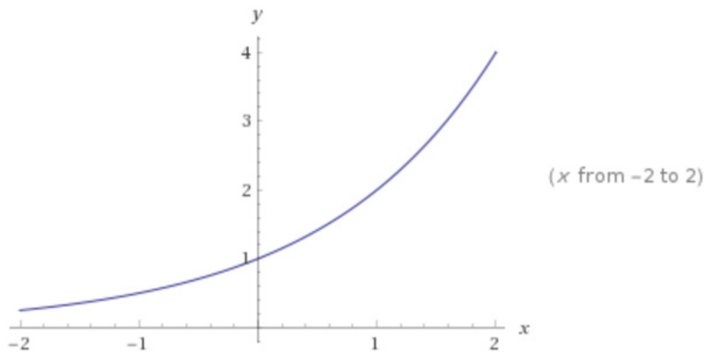

Figure 3: The graph of the curve $y = 2^x$

Anyway, there is no specific reason to study just the function $f(x) = 2^x$. We might also study $g(x) = 5^x$ or $h(x) = 83^(x)$. We can even consider $j(x) = \pi^x$. There is however, a special number e for which the function e^x is granted a special name. Namely, we shall take e to be *Euler's number*.

$$e = 2.71828\ldots$$

The reason e and e^x will is so special will become clear as the course progresses.

The key properties of the function e^x include the fact that $e^{(a+b)} = e^a e^b$ as well as the fact that $(e^a)^b = e^{ab}$