Lecture Notes (Math 90): Week II (Tuesday)

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1 tl;dr: limits and why we care

Let's start with a simple example: Suppose a car is moving at a constant rate of 65 miles/hour along a straight line. Then the position of the car, relative to some fixed original point O, is given by the function $f(t) = (65miles/hour)(t)$.

The idea here is that we are starting with a *velocity* and we are producing a function $f(t)$ determining *position* But that is not the crucial point... What we really want to observe here is this 'recipe' also provides a formula for it's own undoing! *Working in reverse, we should be able to determine determine the velocity of the car from the slope of the position function $f(t)$!*

Now, let's see how well this recipe works in practice: Suppose we wish to compute the rate of change of a function $g(t)$ at some t_0. The above argument indicates that the 'slope' of an arbitrary function $g(x)$ is the right thing to look at mathematically. Since slope is just *rise over run*, we may choose some t_1 slightly bigger then t_0 and consider the quotient

$$\frac{g(t_1) - g(t_0)}{t_1 - t_0}$$

as an approximation of the slope at t_0 and hence an approximation of the rate of change. We would like to believe this quotient gets closer and closer to the rate as t_1 gets closer and close to t_0, but we cannot just take $t_1 = t_0$, since that would entail dividing by zero and the world would cease to exist. How do we proceed?

The answer is to develop the language of limits that ceases to speak of what things *are*, but rather to speak of what things *become*.

1.1 The average rate of change and secant lines

Let's put the above discussion on a more precise footing.

Definition 1. Suppose a function $g(t)$ is defined on an interval $[t_0, t_1]$, then the **average rate of change** of g(t) on $[t_0, t_1]$ is given by the formula

$$\frac{g(t_1) - g(t_0)}{t_1 - t_0} = \frac{\Delta g}{\Delta t}$$

Example. The distance traveled by a falling solid object is given by the formula $y(t) = 16t^2$ (Galileo's law). Find the average rate of change over the interval $[0, 2]$

Answer:

The average rate of change of f is intimately related to another more geometric notion.

Definition 2. Given a function $f(t)$ and points t_0 and t_1 in the domain of f, the secant line is the unique line passing through the points $(t_0, f(t_0)$ and $(t_1, f(t_1))$

Note that, essentially by definition, the slope of the secant line is exactly the average rate of change of $f(t)$ over the interval $[t_0, t_1]$.

Example. Consider the function $f(x) = 2x^2 + 1$. Find the secant line associated with the points $x = 1$ and $x = 1 + h$. What is the slope of this secant line? What happens to the slope as h is made smaller and smaller?

Answer:

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2 Piecewise Functions

Piecewise functions are really excellent for illustrating the importance of limits. By definition, a piecewise function is formed by starting with a domain A and breaking it up into several smaller pieces $A_1, A_2, \ldots$ and specifying a function on each of them.

Example. Suppose we wish to describe a function $f(x)$ that is equal to 1 for $x < 0$ and equal to 0 for $x \geq 0$. So our domain is the real numbers, A_1 is the set of real numbers $x < 0$, and A_2 is the set of real numbers $x \geq 0$. Then we would typically write

$$f(x) = \begin{matrix} 1 & x < 0 \\ 0 & x \geq 0 \end{matrix}$$

Here is another example.

Example. Consider the following piecewise function:

$$f(x) = \begin{matrix} 0 & x < 0 \\ 1 & x = 0 \\ 0 & x > 0 \end{matrix}$$

Consider the graph of this function and imagine yourself walking along it. Or better yet, imagine yourself inside of it! Everything is fine at first, a delightful walk on an early autumn day, with the crisp cold air blowing through your hair. Then a hole suddenly appears beneath your feet. You tumble downwards now, falling into that cold, dark, and unforgiving void. Filled with hungry mutant cockroaches.

This exemplifies the phenomena under consideration. One *expects* $f(x)$ to be equal to 0 based on what $f(x)$ looks like *near* $x = 0$. *What we don't expect is to fall into a hole and get eaten by mutant cockroaches.* How can we make this mathematically precise (sans the cockroaches)?

3 The "good" definition of a limit

One might say there is a *good* definition of a limit, a *better* definition, and a *best* definition of a limit. Without further ado, let us give the *good* definition of a limit.

Definition 3. We say that a function $f(x)$ approaches a limit L at $x = a$ if $f(x)$ gets closer and closer to L as x gets closer and closer to a.

Example. Compute the limit of the function $f(x) = \frac{x^2-9}{x-3}$ as x approaches 3.

Answer:

The key point to observe in the above example is we never actually compute $f(3)$ (which doesn't really make much sense, division by zero and all that). Instead, we study $f(x)$ when x is near 3. And because x is *near*, but not equal to, 3, we can safely cancel the term $(x-3)$.

Example. Consider the following piecewise function:

$$f(x) = \begin{matrix} 3x+1 & x \neq 0 \\ -10 & x = 1 \end{matrix}$$

Graph the function and describe the limit of $f(x)$ as x approaches 1.

Answer:

Let us now give an examples of functions that do not have limit:s at specified points.

Example. Graph the function $f(x) = sin(\frac{1}{x})$. Explain why it does not have a limit as x approaches 0. Do

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Answer:

Another example:

Example. Graph the function

$$f(x) = \begin{matrix} x & x < -3 \\ 1 & x \geq -3 \end{matrix}$$

Explain why this function does not have a limit as x approaches -3.

Answer:

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4 Limit laws

It is always easier to state a few general laws rather then repeatedly work them out in every individual case. To solve a complicated limit, we often wish to break it up into a sequence of more digestable results, the following rules provide us with a method for doing so. Suppose that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, and that c, k are constants. Then we have the following rules.

1. *Rule of Sum:* $\lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L + M$
2. *Rule of Difference:* $\lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c} k \cdot f(x) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c} f(x)g(x) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c} f(x)/g(x) = L/M$
6. *Power Rule:* $\lim_{x\to c} f(x)^n = L^n$

Example. Using the above rules, compute the limit of $f(x) = \sqrt[6]{\frac{x^3+x^2-x-1}{x-1}}$ as x approaches 1.

Answer:
