Lecture Notes (Math 90): Week II (Thursday)

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1 More advanced limits

1.1 Tricks and tools

Let us begun by recalling the limit rules enumerated last class

1. *Rule of Sum:* $\lim_{x\to c}(f(x)+g(x)) = L + M$
2. *Rule of Difference:* $\lim_{x\to c}(f(x)-g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c} k \cdot f(x) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c} f(x)g(x) = L \cdot M$
5. *Quotient Rule (For $M \neq 0$):* $\lim_{x\to c} f(x)/g(x) = L/M$
6. *Power Rule:* $\lim_{x\to c} f(x)^n = L^n$
7. *Root Rule:*

$$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$$

These rules can be effectively used to built up complicated limits by understanding the limits of relatively simple functions. To do this, we will often need to carry out long division of polynomials.

Example. Use polynomial division to divide the polynomial $12 - 12x + x^2 + x^3$ by $(x - 2)$. Then compute the limit

$$\lim_{x\to 2} \frac{12 - 12x + x^2 + x^3}{x - 2}$$

Answer:

Another common example involves computing limits that involve square roots. These can require the use of a special trick.

Example. Compute the limit

$$\lim_{x \to 2} \frac{\sqrt[2]{x+7} - 3}{x - 2}$$

Answer:

1.2 The squeezing theorem

Sometimes, we can evaluate the limit of a function $f(x)$ by placing this function in relation to two other functions.

Theorem 1. *Suppose that* $g(x) \leq f(x) \leq h(x)$ *for all* x *in some open interval containing* c *except possibly at* $x = c$. *Suppose also that* $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$. *Then* $\lim_{x \to c} f(x) = L$

One way to think of the above theorem is to imagine a cave. In such a case, the function $h(x)$ corresponds to the ceiling and $g(x)$ corresponds to the floor. If you (the function $f(x)$) are walking inside the cave, and the floor and the ceiling are closing in, then eventually you , the floor, and the ceiling will all collide at the same place.

The following example indicates how the squeezing theorem may be used in practice.

Example. Suppose that $(4 - x^2) \leq f(x) \leq (4 + x^2)$. Compute $\lim_{x\to 0} f(x)$.

Answer:

Remark. So far, we've mostly done limits involving polynomials, rational functions, and square roots. We will also want to occasionally compute limits involving trigonometric functions. It is not hard to show the following statements

$$\lim_{x\to c} sin(x) = sin(c)$$

$$\lim_{x\to c} cos(x) = cos(c)$$

are true for any real number c. On the other hand, the following limit

$$\lim_{x\to 0} sin(x)/x = 1$$

is non-trivial. You will see a proof of this during a recitation section, but for now you may take it on faith.

Example. Compute $\lim_{x\to 0} \frac{1-3x+cos(x)}{5+cos(x)}$.

Answer:

1.3 Right and left sided limits

So far, we've mostly been concerned with limits concerning both directions of approach. That is, a function $f(x)$ has a limit L if any only it approaches L from both the right and from the left. It is useful to introduce notions for addressing the left/right directions of approach individually.

Definition 1. (Right-hand limits) We say that a function $f(x)$ approaches a limit L on the right at $x = a$ if $f(x)$ gets closer and closer to L as x gets closer and closer to a from the right (That is, $a < x$). We write $\lim_{x\to c^+} f(x)$ for the right-hand limit.

Definition 2. (Left-hand limit) We say that a function $f(x)$ approaches a limit L on the left at $x = a$ if $f(x)$ gets closer and closer to L as x gets closer and closer to a from the left (That is, $x < a$). We write $\lim_{x\to c^-} f(x)$ for the left-hand limit.

Let us recall an example from Tuesday:

Example. Graph the function

$$f(x) = \begin{matrix} x & x < -3 \\ 1 & x \geq -3 \end{matrix}$$

Compute the limits $\lim_{x\to -3^-} f(x)$ and $\lim_{x\to -3^+} f(x)$

Answer:

Example. Compute the limits $\lim_{x\to 0^-} \frac{|x|}{x}$ and $\lim_{x\to 0^+} \frac{|x|}{x}$

Answer: