Lecture Notes (Math 90): Week III (Tuesday)

Alicia Harper

September 19, 2017

1 Continuity

Figure 1: Avoid falling into the trap of reading bad webcomics.

1.1 The definition

In our discussions last week, we talked about limits are useful in understanding the value that the function $f(x)$ approaches as x approaches c. An important point, one that can never really be stressed enough, is that definition of a limit is completely apathetic to the actual value of $f(x)$. So limits will frequently exist *in spite* of disagreeing with the value $f(c)$. That said, the situation where the values do agree, say $\lim_{x\to c} f(x) = f(c)$, is indeed special and demand it's own name.

Definition 1. A function $f(x)$ is said to be continuous at $x = c$ if $\lim_{x \to c} f(x) = f(c)$. We say that a function $f(x)$ is continuous everywhere if this is true for every real number c in the domain of f. We shall say that $f(x)$ is continuous on a closed interval $[a, b]$ is the right-hand limit at $x = a$ is $f(a)$ and the left-hand limit at $x = b$ is $f(b)$.

Definition 2. We shall say $f(x)$ is *discontinuous* at $x = c$ if either the limit $\lim_{x \to c} f(x)$ does not exist or if it does not agree with the value of $f(c)$. In the latter case, where the limit exists, we shall say that $f(x)$ has a removable discontinuity at c

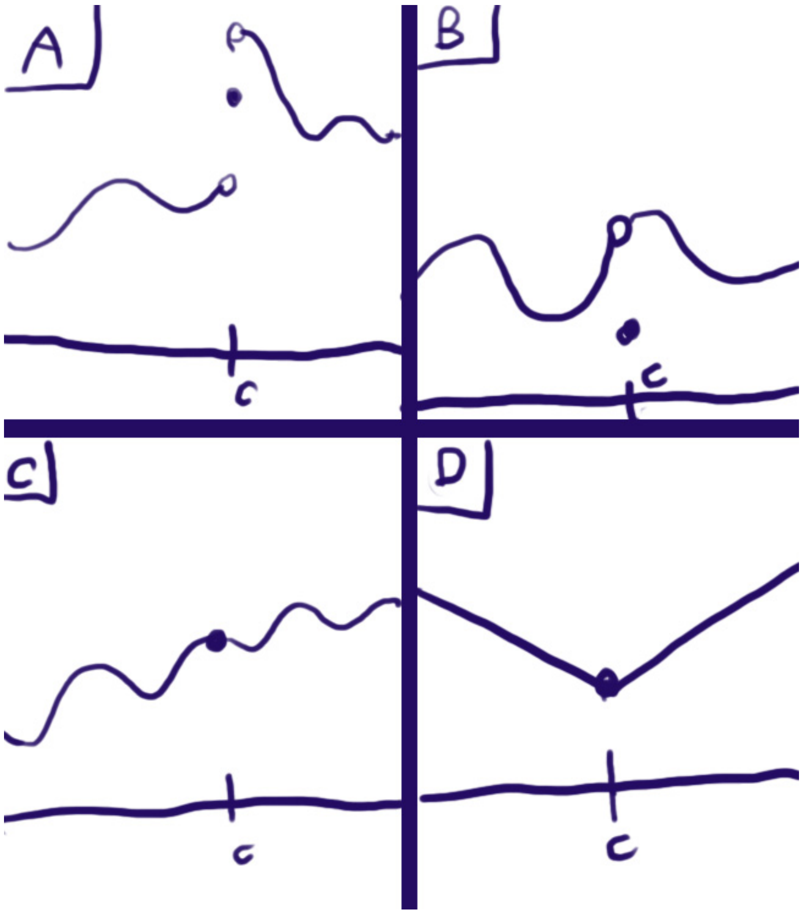


Figure 2: In (A), the function does not have a limit as $x \to c$. In (B), the function does have a limit as $x \to c$, but it is not continuous, so it has a removable discontinuity. Both the (C) and (D) have limits and are continuous as $x \to c$.

In some sense, we've been using a notion of continuity all along: Whenever we evaluate a limit like $\lim_{x \to 2} x^3 + 9$ by substitution, we are really taking advantage of the fact that polynomials are always continuous. Continuity obeys a set of rules similar to the ones enumerated for limits.

Remark. Suppose that $f(x)$ and $g(x)$ are continuous at $x = c$ and k is a constant, then the following functions are continuous at $x = c$.

1. $f(x) + g(x)$

2. $f(x) - g(x)$

3. $k \times f(x)$

4. $f(x) \cdot g(x)$

5. $f(x)/g(x)$ if $g(c) \neq 0$

6. $f(x)^n$

7. $\sqrt[n]{f(x)}$

These rules are useful for rigorously establishing the above claim that all polynomials are continuous, since we need only verify it for constants and $f(x) = x$. This is not an introduction to analysis, so we will not take that route. Two more useful rules that deserves to be separated out from the herd is as follows:

Remark. Suppose that $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $f(c)$, then $(g \circ f)(x)$ is continuous at $x = c$

Remark. Suppose that $f(x)$ is continuous at $x = c$ and has an inverse $f^{-1}(x)$ defined near $x = f(c)$. Then $f^{-1}(x)$ is continuous at $x = f(c)$

Finally, we state one of the most remarkable theorems about continuity:

Remark. Suppose that $f(x)$ is continuous on the interval $[a, b]$ and $f(a) \leq c \leq f(b)$, then there exists a number y in the closed interval such that $f(y) = c$.

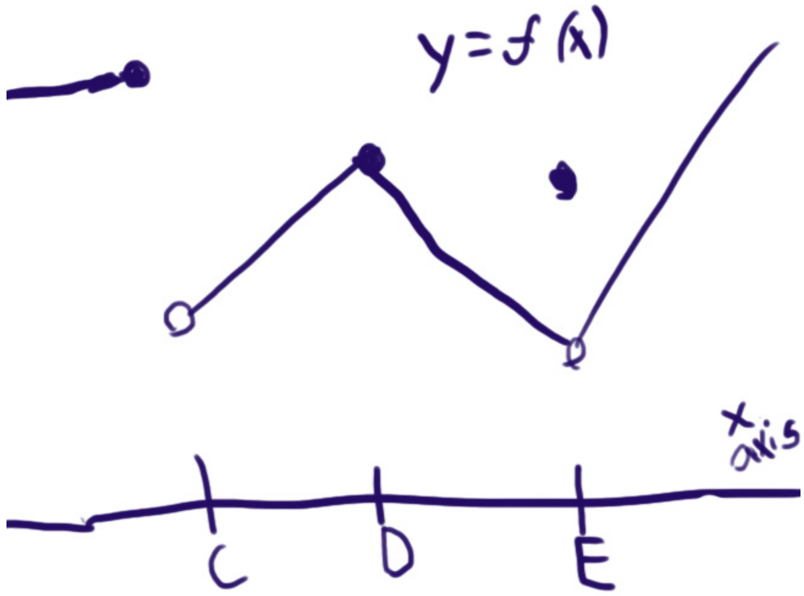
2 To infinity and beyond: Infinite limits, and vertical/horizontal asymptotes

Up until now, we have constrained ourselves to addressing limits $x \to c$ where c is a real number. We can also take c to be either ∞ or $-\infty$ and, as we shall see, this sometimes makes sense and tells us meaningful qualitative information about the graph of $f(x)$.

Definition 3. We shall say $f(x)$ has a horizontal asymptote if either $\lim_{x\to\infty} f(x) = a$ or $\lim_{x\to\infty} f(x) = b$. Note that a function may have two distinct horizontal asymptotes corresponding to ∞ and $-\infty$.

Definition 4. We shall say $f(x)$ has a vertical asymptote if any of the following limits hold $\lim_{x\to c^+} f(x) = \infty$, $\lim_{x\to c^+} f(x) = -\infty$,$\lim_{x\to c^-} f(x) = \infty$, and $\lim_{x\to c^-} f(x) = -\infty$. Note that a function may have two distinct horizontal asymptotes corresponding to ∞ and $-\infty$.

Example 1. At which of the marked points does the (poorly drawn) function below have a limit? When is it continuous?



Example 2. Let floor(x) be the largest integer that is smaller than x. So for example, $floor(\pi) = 3$. Graph floor(x). When is the function continuous?

Example 3. Explain why the functions $f(x) = \sqrt[3]{x^9 + 5x + 3}$ and $g(x) = \frac{(x-3)cos(x)}{x^2-9}$ are both continuous on their domains of definition.

Example 4. Explain why $sin^{-1}(\sqrt[2]{x^{100} + x^{50} + 1})$ is continuous on it's domain of definition.

Example 5. Using the intermediate value theorem, show that a chicken crossing the road always must walk over the median. (Challenge: Show that there are always two points on the opposite side of the earth with the same temperature)

Example 6. Show that the function $x^4 + 7x - 1$ has a root in the closed interval $[0, 2]$. You do not need to calculate the root.

Example 7. Figure out what the limits $\lim_{x\to\infty} \frac{1}{1+x}$ and $\lim_{x\to-\infty} \frac{1}{1+x}$ are by experimenting with values. Do the same for $\lim_{x\to\infty} \frac{x}{1+x}$.

Example 8. Find the horizontal asymptotes of

$$\frac{3x^2 + 5x + 1}{|x|^2 + 1}$$

$$\sin(\frac{3}{x^4 + 1})$$

$$\tan(\frac{\pi x + 4}{4x + 9})$$

Example 9. Find the vertical asymptotes of

$$\frac{x+1}{x^2+2x+1}$$

Example 10. Find all vertical asymptotes of

$$\frac{3}{1+cos(x)}$$