Lecture Notes (Math 90): Week III (Thursday)

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1 Infinite limits, part II

1.1 Vertical asymptotes

To begin, let us recall the definition of horizontal limits:

Definition 1. We shall say $f(x)$ has a *horizontal* asymptote if either $\lim_{x\to\infty} f(x) = a$ or $\lim_{x\to\infty} f(x) = b$. Note that a function may have two distinct horizontal asymptotes corresponding to ∞ and $-\infty$.

Horizontal limits, as discussed on Tuesday, concern the limit of $f(x)$ as x goes to ∞. *Vertical* limits, defined blow, instead concerns the possibility of $f(x)$ going to $\pm\infty$ as x goes towards some real number c. Geometrically, this means that the graph of $y = f(x)$ tends towards the vertical line $x = c$.

Definition 2. We shall say $f(x)$ has a *vertical asymptote* if any of the following limits hold

$$\lim_{x\to c^+} f(x) = \infty$$

$$\lim_{x\to c^+} f(x) = -\infty$$

$$\lim_{x\to c^-} f(x) = \infty$$

$$\lim_{x\to c^-} f(x) = -\infty$$

Note that a function may have two distinct vertical asymptotes corresponding to the left and right direction of approach.

There is one other notion of an asymptote that that was must introduce. Intuitively, a function $f(x)$ has an *oblique* asymptote if it behaves like the function $y = ax + b$ as x goes to either $\pm\infty$. More precisely, the goal is to find the line $y = ax+b$ such that one of the following two limits hold:

$$\lim x \to \infty \frac{f(x)}{ax+b} = 1$$

$$\lim x \to -\infty \frac{f(x)}{ax+b} = 1$$

1.2 Derivatives

Now that we have introduced limits, we may formally introduce the notion of a *derivative.* Recall the prototypical problem: One has an object moving along a line with position at a time t given by the function $f(t)$, at some time t_0 how fast is the object moving? Using the average rate of change formula on small intervals, we arrived at the following formula (and, also, the definition of the derivative).

Definition 3. A function $f(x)$ is said to be *differentiable* at $x = c$ if $f(x)$ is continuous and the following limit exists

$$\lim h \to 0 \frac{f(c+h) - f(c)}{h}$$

We denote the value of the above limit by $f'(c)$. $f'(c)$ is referred to as the *derivative* of $f(x)$ at $x = c$.

In the above definition, we are thinking of $f'(c)$ as just the value of a certain limit. However, as the notation might suggest, we might also regard $f'(c)$ as the value of a function f'. Thus, we can think of the derivative as a function: It takes in a specific point (or time) and returns the rate of change of the function at that specific point (or time).

Note also that it is important to know this definition and to know how to work with it. Soon you will be given rules that allow you to efficiently compute derivatives, we may (and probably will) ask you to compute at least one derivative directly from definition on an exam.

Example 1. Find the horizontal asymptotes of

$$\sin(\frac{3}{x^4+1})$$

$$\tan(\frac{\pi x+4}{4x+9})$$

Example 2. Find all vertical asymptotes of

$$\frac{2x+5}{x^2-1}$$

Example 3. Find all vertical asymptotes of

$$\frac{3}{1 + cos(x)}$$

Example 4. Find the vertical asymptotes of

$$\frac{x + 1}{x^2 + 2x + 1}$$

Example 5. Find the oblique asymptotes of

$$\frac{2x^2 + 5}{9x + 1}$$

$$\frac{9x^3 + x + 5}{2x^2 + x + 1}$$

Example 6. Compute the derivative of $2x^2$ at $x = 5$ from the definition. Now compute the derivative of $2x^2$ at $x = 21$. What about $x = 100$?

Example 7. Compute the derivative of $f(x) = \frac{2}{x}$.

Example 8. Compute the derivative of $f(x) = sqrt[2](x)$ for $x > 0$.

Example 9. Compute the derivative of $f(x) = \frac{2}{x}$.

Example 10. Compute the derivative of $f(x) = \sqrt[2]{(x)}$ for $x > 0$.

Example 11. Compute the derivative of $f(x) = \frac{2}{x}$. Also compute the derivative of $f(x) = \sqrt[2]{(x)}$ for $x > 0$.