Lecture Notes (Math 90): Week IV (Tuesday)

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1 Derivatives

1.1 The definition,

Now that we have introduced limits, we may formally introduce the notion of a *derivative.* Recall the prototypical problem: One has an object moving along a line with position at a time t given by the function $f(t)$, at some time t_0 how fast is the object moving? Using the average rate of change formula on small intervals, we arrived at the following formula (and, also, the definition of the derivative).

Definition 1. A function $f(x)$ is said to be *differentiable* at $x = c$ if $f(x)$ is continuous and the following limit exists

$$\lim h \to 0 \frac{f(c+h) - f(c)}{h}$$

We denote the value of the above limit by $f'(c)$ or by $\frac{df}{dx}(c)$. $f'(c)$ is referred to as the *derivative* of $f(x)$ at $x = c$.

In the above definition, we are thinking of $f'(c)$ as just the value of a certain limit. However, as the notation might suggest, we might also regard $f'(c)$ as the value of a function f'. Thus, we can think of the derivative as a function: It takes in a specific point (or time) and returns the rate of change of the function at that specific point (or time).

Note also that it is important to know this definition and to know how to work with it. Soon you will be given rules that allow you to efficiently compute derivatives, we may (and probably will) ask you to compute at least one derivative directly from definition on an exam.

1.2 Rules and techniques

The following list provides a set of tools to make the calculation of derivatives ultimately routine. Though the most important technique - the chain rule - will have to wait until next week, already these tools allow us to evaluate a very large number of examples. Differentiation Rules:

1. $(f+g)'(x) = f'(x) + g'(x)$ (The rule of sum)

2. $\frac{d}{dx}(k \cdot f(x)) = k \cdot f'(x)$ for all constants k. (The constant rule)

3. $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$ (The product rule)

4. $\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)}^2$ (The product rule)

5. $\frac{d}{dx}[x^c] = cx^{c-1}$ for *any* real number c. (The generalized power rule)

We shall also need to state the following remark:

Remark. The derivative of e^x is just e^x.

Example 1. Compute the derivative of $f(x) = \frac{2}{x}$.

Example 2. Compute the derivative of $f(x) = sqrt[2](x)$ for $x > 0$.

Example 3. Show that the function $f(x) = |x|$ does not have a derivative at $x = 0$.

Example 4. (The constant rule): Let k be a constant. Show that the derivative of $k \cdot f(x)$ is $k \cdot f'(x)$.

Example 5. (The rule of sum): Show that the derivative of $(f+g)(x)$ is just $f'(x)+g'(x)$ [The sum of the derivatives].

Example 6. (The Product Rule): Show that the derivative of $f(x)g(x)$ is $f(x)g'(x)+f'(x)g(x)$ formally and by way of picture.

Example 7. (The Power Rule for integers): Let $f(x) = x$. Compute $f'(x)$, and use the product rule to compute the derivative of x^n more generally.

Example 8. Compute the derivative of $x^4 + 9x^3 + 2x^2 + 1$ at $x = 0$.

Example 9. Use the generalized power rule to evaluate the derivative of $f(x) = x^{1/5} + x^2 + 10x^{4/18}$ wherever it is defined.

Example 10. Does the curve defined by the equation $y = \frac{1}{3}x^3 - x$ have any horizontal tangencies? If so, where?

Example 11. Compute the derivative of $f(x) = \frac{1-x^2}{2+e^x}$

Example 12. Compute the derivative of $f(x) = \frac{3x^2+5x+1}{x^3+1}$