Lecture Notes (Math 90): Week IV (Thursday)

Alicia Harper

September 29, 2017

1 Quotient rule

Today, we shall introduce our penultimate rule for computing derivatives. It is an extremely useful one in practice, and will allow us to deal with a fairly wide variety of examples.

Theorem 1. *Suppose $f(x)$ and $g(x)$ are differentiable at $x = c$ and $g(c) \neq 0$, then $\frac{f(x)}{g(x)}$ is differentiable at $x = c$ and it's derivative is given by the following formula:*

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

We shall use this to solve many of the examples below. But before we do, we need to introduce the derivatives of the special functions:

2 Derivatives of special functions

The following is a list of derivatives of standard functions. Commit this list to heart! And please *do not forget* a single line until the first day of winter break! T

1. The derivative of e^x is e^x
2. The derivative of $sin(x)$ is $cos(x)$
3. The derivative of $cos(x)$ is $-sin(x)$
4. The derivative of $tan(x)$ is $sec^2(x)$
5. The derivative of $sec(x)$ is $sec(x)tan(x)$
6. The derivative of $cot(x)$ is $-csc^2(x)$
7. The derivative of $csc(x)$ is $-csc(x)cot(x)$

You should memorize all of these, but you should also note that the last four of the above rules are *essentially redundant* and in fact follows from rules for $sin(x)$ and $cos(x)$ by the quotient rule and the standard trigonometric identities.

3 Higher derivatives, velocity, and acceleration

One of our original motivations for studying derivatives was the fact that the velocity of a moving object (say whose position is given by a function $x(t)$), is exactly the derivative $x'(t)$. But to better understand the physics of moving objects (let alone the underlying mathematics), it will serve us well to consider the *higher derivatives* of $x(t)$.

To motivate the importance of higher derivatives: Imagine you are on a roller coaster - the cart begins its upwards ascent on inverted V, and then rushes downward. Then, in that moment, you feel the weight of your body thrown back against the chair. That feeling of force is not related with speed, but instead with acceleration. Acceleration is the *second derivative* of position. Just as we may form the derivative of $f(x)$, we may also form the derivative of $f'(x)$. This derivative is often denoted as $f''(x) = f(x)$.

With the notion of the *second derivative* in hand, we can now state Newton's famous law:

$$F(t) = m \cdot x''(t)$$

where m is the mass of an object, $x(t)$ is the function giving the position of an object at time t, and $F(t)$ is the *force* acting on the object.

We also note, though it is somewhat less important, that the third derivative $x'''(t)$ is called *jerk*.

One othe r example of derivatives in applications is the marginal cost of production: Let $c(x)$ be the cost of producing x units, then the derivative $c'(x)$ is the *marginal cost of production*.

Example 1. Compute the derivative of $f(x) = \frac{1-x^2}{2+e^x}$

Example 2. Compute the derivative of $f(x) = \frac{3x^2+5x+1}{x^3+1}$ wherever it is defined.

Example 3. Compute the derivative of $f(x) = \frac{sin(x)}{cos(x)+sin(x)}$ wherever it is defined.

Example 4. Find the tangent line to the graph of the function $f(x) = \frac{sin(x)}{cos(x)+sin(x)}$ at $x = \pi$.

Example 5. Compute the limit $\lim_{\theta \to \pi/6} \frac{sin(\theta) - \frac{1}{2}}{\theta - \pi/6}$

Example 6. At time $t \geq 0$, the velocity of a body moving along the horizontal s-axis is $v = t^2 - 4t + 3$.

1. Find the body's acceleration each time the velocity is zero.
2. When is the body moving forward? Backward?
3. When is the body's velocity increasing/decreasing?

Example 7. A 45-caliber bullet shot straight up from the surface of the moon would reach a height of $s = 832 - 2.6t^2$ feet after t seconds. On earth, in the absence of air, its height would be $s = 832 - 16t^2$ feet after t seconds. How long will the bullet be aloft in each case? How high will the bullet go?

Example 8. Using only the derivative rules for sine and cosine, compute the derivatives of $tan(x)$ and $cot(x)$.

+]

Example 9. Compute the second and fourth derivatives of sin(x) and cos(x).

Example 10. Compute the second derivative of $f(x) = \frac{sin(x)+5}{e^x}$.