# Lecture Notes (Math 90): Week VI (Tuesday)

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Today, we continue our study of the chain rule and it's consequences. One area where the chain rule sheds new light is in the study of inverse functions. Suppose that $f(x)$ is invertible, so that there is a function $f^{-1}(x)$ with

$$f(f^{-1}(x)) = x$$

$$f^{-1}(f(x)) = x$$

We know that the graph of $f^{-1}(x)$ is essentially a flipped version of the graph of $f(x)$.

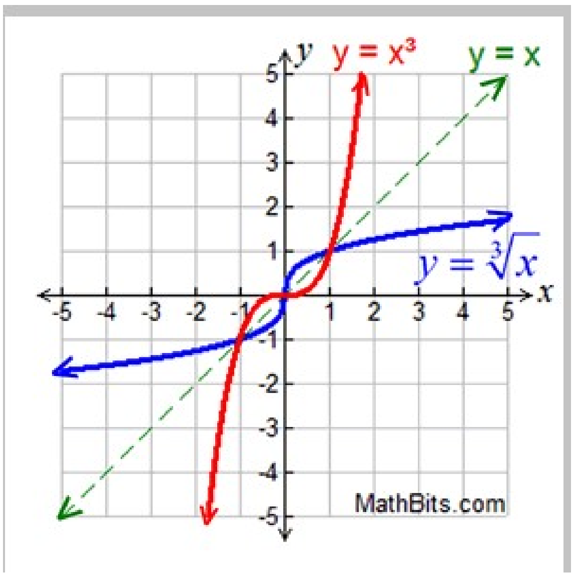


Figure 1: $\sqrt[2]{x}$ is the inverse function $f(x) = x^2$

So, it is reasonable to expect there existence of a technique for computing $\frac{df^{-1}}{dx}(x)$ from $f(x)$ and $f'(x)$. It turns out that there does exist such a technique, and we may use it to understand the derivatives of logarithms and inverse trig functions.

## 1 Inverse Functions

In order to derive the inverse function rule for $f^{-1}(x)$, we shall make use of the equation $f^{-1}(f(x)) = x$. Since the left-hand side of the equation is a composition of two functions, we might seek to differentiate both sides and apply the chain rule. This yields the equations

$$\frac{df^{-1}}{dx}(f(x))f'(x) = 1$$

$$\frac{df^{-1}}{dx}(f(x)) = \frac{1}{f'(x)}$$

So, by substituting in $f^{-1}(x)$, we finally obtain

$$\frac{df^{-1}}{dx}(f(f^{-1}(x))) = \frac{1}{f'(f^{-1}(x))}$$

$$\frac{df^{-1}}{dx}(x)) = \frac{1}{f'(f^{-1}(x))}$$

Let us now crystallize this rule in the following theorem:

**Theorem 1.** *If $f(x)$ has an inverse function $f^{-1}(x)$ on an interval $(a, b)$, and $f(x)$ is differentiable with derivative $f'(x)$ never vanishing on the interval $(a, b)$, then $f^{-1}(x)$ is also differentiable and it's derivative is given by the following formula:*

$$\frac{df^{-1}}{dx}(x)) = \frac{1}{f'(f^{-1}(x))}$$

**Example 1.** Use the inverse function theorem to compute the derivative of $f(x) = \sqrt[2]{x}$.

**Example 2.** Let $f(x) = x^3 - 2$. Note that $f(2) = 6$. Find the value of $\frac{df^{-1}}{dx}(6))$.

**Example 3.** Use the inverse function theorem to compute the derivatives of $arcsin(x)$, $arccos(x)$, and $arctan(x)$.

**Example 4.** Use the inverse function theorem to compute the derivative of $f(x) = ln(x)$.

**Example 5.** More generally, compute the derivatives of $e^a x$ and $log_a(x)$.

**Example 6.** How does knowing the derivative of $ln(x)$ help us more generally? Use logarithmic differentiation to compute the derivative of $y = \frac{(x^2+1)(x+1)^{\frac{1}{2}}}{x-1}$

**Example 7.** How does the exponential give us a definition of exponentiation $x^a$ for *irrational* $a$.
for Differentiate $f(x) = x^x$

**Example 8.** Show that $\lim_{x \to 0}(1 + x)^{1/x} = e$