Lecture Notes (Math 90): Week VIII (Tuesday)

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Extreme Values

1 The Story

On Tuesday, we explored some of the first applications of calculus, the related rates problems. Although these provide great applications for the general theory, they aren't exactly the most common problems one finds in applications.

The majority of real life applications of differential calculus are focused on maximizing and minimizing a certain function $f(x)$. For example, suppose a company is selling a product and they believe that the potential profits obey some function $P(x)$ where x is the quantity of products produced. If they are correct, then it makes sense to try and find the *maximum value* of the function $P(X)$.

As a second example, suppose at object is moving up and down with height function $x(t)$. When is the object stationary? If you imagine a pendulum, then the weight of the pendulum is stationary precisely at the point where it is highest on it's arc. Such a problem also amounts to finding the maximum value of the function $x(t)$.

For our final example, neural networks are composed of single neurons. For us, mathematically we can model a single neuron with two inputs by the function $f(x, y, w_1, w_2) = x \cdot w_1 + y \cdot w_2$. In machine learning, one trains neurons by feeding them *training data* of desired input/output values, One then tries to choose weights w_1 and w_2 such that the function f closely approximates this data. This is a minimization problem, and the techniques we shall develop today do indeed form some of the theoretical background necessary for machine learning.

2 The Strategy

The above examples illustrate the importance of finding the maximums and minimums of functions $f(x)$. But how exactly do we find them? To start, consider the following theorem:

Theorem 1. *If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers $x_max, x_min \in [a, b]$ such that $f(x_min) \leq f(x)$ and $f(x) \leq f(x_max)$ for all $x \in [a, b]$.*

The above theorem relies crucially on the fact that f is continuous. It fails for non-continuous functions as the following example shows:

The above theorem is also not constructive: It tells us that x_min and x_max exist, but it does not tell us how to find them. In order to do that, we must assume the function $f(x)$ is differentiable.

Definition 1. A function $f(x)$ is said to have a *local maximum* at x_0 if $f(x) \leq f(x_0)$ for all x that are close to x_0.

Definition 2. A function $f(x)$ is said to have a *local minimum* at x_0 if $f(x_0) \leq f(x)$ for all x that are close to x_0.

Theorem 2. *If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then*

$$f'(c) = 0$$

With the above theorems in mind, we can outline the following general strategy for finding the local maximums and minimums of a differentiable function defined on a closed interval.

1. First be sure you have correctly identified the function $f(x)$ and the interval [a,b]

2. Find all points x_i in $[a, b]$ such that $f'(x) = 0$.

3. Evaluate the function $f(x)$ at each of the x_i *and* the points a and b.

4. The largest of these values is the maximum value and the smallest is the minimum.

Examples

Example 1. Find the maximum and minimum values of $f(x) = x^3 + 3x + 7$ on $[-2, 2]$

1. Identify the function and the interval

2. Set the first derivative to zero

3. Test the function on all the points

4. Identify which is the largest and the smallest.

Example 2. Find the maximum value of $f(x) = 10x(2 - ln(x))$ on the interval $[1, e^2]$.

1. Identify the function and the interval

2. Set the first derivative to zero

3. Test the function on all the points

4. Identify which is the largest and the smallest.

Example 3. Find the maximum value of $f(x) = -\sqrt[2]{5 - x^2}$ on $[-\sqrt[2]{5}, 0]$

1. Identify the function and the interval

2. Set the first derivative to zero

3. Test the function on all the points

4. Identify which is the largest and the smallest.

Example 4. A thrown object has position function $x(t) = -\frac{-g}{2}t^2 - 5$. When does the object reach it's highest point?

1. Identify the function and the interval

2. Set the first derivative to zero

3. Test the function on all the points

4. Identify which is the largest and the smallest.

Example 5. Suppose we are allowed to use 20 square feet of cardboard to make an open box with a square base. Our box will have height h, and the base will have length and width equal to x. Finding the largest box one could make.

1. Identify the function and the interval

2. Set the first derivative to zero

3. Test the function on all the points

4. Identify which is the largest and the smallest.