Lecture Notes (Math 90): Week IX (Tuesday)

Alicia Harper

November 13, 2017

Existence theorems and monotonic functions

1 The Story

1.1 Two more existence theorems!

Recall the intermediate value theorem:

Theorem 1. *If a function $f(x)$ is continuous on a closed interval $[a, b]$ and d is between the values of $f(a)$ and $f(b)$, then there exists some $c \in [a, b]$ such that $f(c) = d$.*

As previously discussed, this is an example of an *existence theorem*. Existence theorems are somewhat of a mixed bag: they tell us that something interesting is happening... somewhere, but they do not tell us where it is happening! Despite this flaw, existence theorems are sometimes useful in showing that what we expect to be true is actually true. The following examples will hopefully make this more clear. aa

Example 1. Suppose a hiker is traveling over a mountain range, and walking *up* a one dimensional mountain at sunrise, and traveling *down* another one at the time sun sunsets. Must the hiker eventually walk over completely level ground?

Assuming the mountain is nice (by this, i mean it has no spikey peak as in the figure 1.b above), then answer is yes: Since he went from traveling *upwards* at sunrise to traveling *downwards* at sunset, his velocity in vertical direction must have been zero at some intermediate time. Let's crystallize idea this into a theorem:

Theorem 2. *Suppose that $y = f(x)$ is continuous over the closed interval $[a.b]$ and differentiable at every point of its interior $[a, b]$. If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.*

Example 2. Suppose a car travels one 180 miles on a straight highway in 2 hours. The speed limit for the highway is always 85 miles per hour. Must the driver have violated the speed limit at some point?

I0f the driver was driving at or below the speed limit, then we know that he would have traveled at most 170 miles over the two hour period. So, barring the use of teleportation apparatus, he must have been driving above the speed limit at some point. In fact, we can say slightly more: There was at least one point during the two drive where he was driving exactly 90 miles per hour. Think of it like this: If was always driving more than 90 miles per hour, he would have traveled a total distance strictly greater than 180 miles. If he was always driving less than 90 miles per hour, we have traveled a total distance strictly less than 180 miles. As we did above, let us turn this example into an actual theorem:

Theorem 3. *Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

1.2 Monotonic Functions

Recall that we say a function is *increasing* if $f(a) > f(b)$ for all $a > b$. Likewise, we say that a function f is *decreasing* if if $f(a) < f(b)$ for all $a > b$. For differentiable functions, we have a nice way to go about relating these notions to the values of derivatives.

Proposition 1. *Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).*

1. *If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.*
2. *If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.*

Proof. Suppose that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) >$ for $x \in (a, b)$, then the Mean Value Theorem tells us that for any $x_1 > x_2$ in $[a, b]$, we have

$$f(x_1) - f(x_2) = f'(c)(x_1 - x_2)$$

for some c between x_1 and x_2. The proof for the $f'(x) < 0$ case is similar.

□

This provides a useful tool for the analysis of critical points.

Proposition 2. *Suppose that c is a critical point of a continuous and differentiable function f, then:*

1. *If f' changes from a negative to a positive at c, then f has a local minimum at c.*
2. *If f' changes from a positive to a negative at c, then f has a local maximum at c.*
3. *If f' does not change sign, then c is not a local extremum.*

2 The Strategy

The results from before allow us to improve our strategy for studying the behavior of focus from last time. We include a new step

1. First be sure you have correctly identified the function $f(x)$ and the interval $[a, b]$.
2. Find all points x_i in $[a, b]$ such that $f'(x) = 0$. Also consider the points where $f(x)$ is not differentiable, as well as the points a and b.
3. To find the local extremums, evaluate the sign of $f'(x)$ on the intervals between the points found in the previous step.
4. Evaluate the function $f(x)$ at each of the x_i *and* the points a and b.
5. The largest of these values is the maximum value and the smallest is the minimum.

Examples

Example 3. Show that if $f'(x) = 0$, then $f(x)$ is a constant function.

Example 4. Show that the function $x^4 + 3x + 1$ has exactly one zero in the interval $[-2, -1]$.

Examples

Example 5. Find the critical points of $f(x) = x3 - 12x - 5$. Use them to find the open intervals on which f is increasing and on which f is decreasing.

Example 6. Show that the function $x^4 + 3x + 1$ has exactly one zero in the interval $[-2, -1]$.