Lecture Notes (Math 90): Week IX (Thursday)

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Monotonic functions/Convexity and Concavity

1 The Story

1.1 Monotonic Functions

Recall that we say a function is *increasing* if $f(a) > f(b)$ for all $a > b$. Likewise, we say that a function f is *decreasing* if if $f(a) < f(b)$ for all $a > b$. For differentiable functions, we have a nice way to go about relating these notions to the values of derivatives.

Proposition 1. *Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).*

1. *If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.*
2. *If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.*

Proof. Suppose that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) >$ for $x \in (a, b)$, then the Mean Value Theorem tells us that for any $x_1 > x_2$ in $[a, b]$, we have

$$f(x_1) - f(x_2) = f'(c)(x_1 - x_2)$$

for some c between x_1 and x_2. The proof for the $f'(x) < 0$ case is similar.

□

This provides a useful tool for the analysis of critical points.

Proposition 2. *Suppose that c is a critical point of a continuous and differentiable function $f(x)$, then:*

1. *If f' changes from a negative to a positive at c, then f has a local minimum at c.*
2. *If f' changes from a positive to a negative at c, then f has a local maximum at c.*
3. *If f' does not change sign, then c is not a local extremum.*

Concavity and convexity explain are related to how the derivative of $f(x)$ changes as x varies. You might picture them as a describing how the tangent line to the curve $y = f(x)$ 'turns'.

Definition 1. 1. We say that $f(x)$ is concave up on an interval I where $f'(x)$ is increasing.

2. We say that $f(x)$ is concave down on an interval I where $f'(x)$ is decreasing.

Sometimes a function, like say $f(x) = x^3$, changes from concave down to concave up to concave down at a certain point. We will need to give a name to such points.

Definition 2. We say that c is a point of inflection for $f(x)$ if $f(x)$ has a tangent line at c and the concavity of f changes at c.

Since concavity is related to the fact that $f'(x)$ is either increasing or decreasing, the results of the previous section tells us that:

Proposition 3. *If $f(x)$ has an inflection point at c, then either $f''(c) = 0$ or $f''(x)$ does not exist.*

Finally, we may state the second derivative test.

Proposition 4. *Suppose that $f'(c) = 0$ then:*

1. *If $f''(c) > 0$ then f has a local minimum at c.*

2. *If $f''(c) < 0$ then f has a local maximum at c.*

3. *If $f''(c) = 0$ then the test fails and one must the first derivative test.*

2 The Strategy

The results from above allow us to improve our general strategy for studying the behavior of functions. We recall the general setup from last time and we include two new steps:

1. First be sure you have correctly identified the function $f(x)$ and the interval $[a, b]$.

2. Find the critical points x_i in $[a, b]$: Conisder all points x such that $f'(x) = 0$, togetherwith the points where $f'(x)$ is not defined, as well as the points a and b.

3. Find the inflections points x_i in $[a, b]$: Conisder all points x such that $f''(x) = 0$, together with the points where $f'(x)$ is not differentiable.

4. To find the local extremums, evaluate the sign of $f'(x)$ on the intervals between the points found in the previous step. OR, apply the second derivative test

5. Using the inflection points, find the regions where $f(x)$ is a convex or concave.

6. Evaluate the function $f(x)$ at each of the x_i *and* the points a and b.

7. The largest of these values is the absolute maximum value and the smallest is the absolute minimum.

8. Make a very loose sketch of the function based on all of the above information.

Example 1. Find the critical points of $f(x) = x^3 - 12x - 5$. Apply the general strategy to obtain a sketch of the curve.

1. Identify the function and the interval. Compute the first and second derivative.

2. Critical points: Find all points where $f'(x) = 0$. Also note any points of non-differentiability if they exist.

3. Inflection points: Find all points where $f''(x) = 0$. Also note any points of non-differentiability if they exist.

4. Test the derivative of the function between all of the the above points (Including the end points, if any.)

5. Test the function on all critical points

6. Find the regions where $f(x)$ is convex or concave.

7. Identify the absolute maximum and minimum.

8. Graph the function.

Example 2. Consider the function $f(x) = x^{1/3}(x-4)$. Apply the general strategy to obtain a sketch of the curve.

1. Identify the function and the interval. Compute the first and second derivative.

2. Critical points: Find all points where $f'(x) = 0$. Also note any points of non-differentiability if they exist.

3. Inflection points: Find all points where $f''(x) = 0$. Also note any points of non-differentiability if they exist.

4. Test the derivative of the function between all of the the above points (Including the end points, if any.)

5. Test the function on all critical points

6. Find the regions where $f(x)$ is convex or concave.

7. Identify the absolute maximum and minimum.

8. Graph the function.

Example 3. Find the critical points of $f(x) = xln(x)$ on the whole real line. Apply the general strategy to obtain a sketch of the curve.

1. Identify the function and the interval. Compute the first and second derivative.

2. Critical points: Find all points where $f'(x) = 0$. Also note any points of non-differentiability if they exist.

3. Inflection points: Find all points where $f''(x) = 0$. Also note any points of non-differentiability if they exist.

4. Test the derivative of the function between all of the the above points (Including the end points, if any.)

5. Test the function on all critical points

6. Find the regions where $f(x)$ is convex or concave.

7. Identify the absolute maximum and minimum.

8. Graph the function.

Example 4. Consider the function $f(x) = \sqrt[2]{3}cos(x)+sin(x)$ on $[0, 2\pi]$. Apply the general strategy to obtain a sketch of the curve..

1. Identify the function and the interval. Compute the first and second derivative.

2. Critical points: Find all points where $f'(x) = 0$. Also note any points of non-differentiability if they exist.

3. Inflection points: Find all points where $f''(x) = 0$. Also note any points of non-differentiability if they exist.

4. Test the derivative of the function between all of the the above points (Including the end points, if any.)

5. Test the function on all critical points

6. Find the regions where $f(x)$ is convex or concave.

7. Identify the absolute maximum and minimum.

8. Graph the function.

Example 5. Consider the function $f(x) = \frac{(1+x)^2}{(1+x^2)}$. Apply the general strategy to obtain a sketch of the curve.

1. Identify the function and the interval. Compute the first and second derivative.

2. Critical points: Find all points where $f'(x) = 0$. Also note any points of non-differentiability if they exist.

3. Inflection points: Find all points where $f''(x) = 0$. Also note any points of non-differentiability if they exist.

4. Test the derivative of the function between all of the the above points (Including the end points, if any.)

5. Test the function on all critical points

6. Find the regions where $f(x)$ is convex or concave.

7. Identify the absolute maximum and minimum.

8. Graph the function.