Lecture Notes (Math 90): Week X (Tuesday)

Alicia Harper

November 13, 2017

L'Hopital's rule

1 The Story

So far, the overall logic of this course has been

1. Define the limits of functions
2. Use limits to define continuity
3. Use limits *and* continuity to define derivatives

One of the amazing facts about calculus is that once we define derivatives, we can actually use them to study and *compute* difficult limits. For example, we might consider the limit:

$$\lim_{x\to 2} \frac{-48+52x-18x^2+2x^3}{6-3x-6x^2+3x^3}$$

It's easy to check that the numerator and denominator are both equal to zero at $x = 2$, and thus we cannot evaluate these limit using a naive substitution. According to the techniques we've learned so far in this class, the best way to approach this problem would probably be to attempt to factor both the numerator and the denominator. Unfortunately, both of the polynomials are of degree three, and that would require a lot of work! To do something simpler, let us write

$$P(X) = -48+52x-18x^2+2x^3$$

$$Q(X) = 6-3x-6x^2+3x^3$$

So that $P(2) = 0$ and $Q(2) = 0$. Because both of these quantities are zero, we can do something that is really quite cute:

$$\lim_{x\to 2} \frac{-48+52x-18x^2+2x^3}{6-3x-6x^2+3x^3} = \lim_{x\to 2} \frac{P(x)}{Q(x)} = \lim_{x\to 2} \frac{P(x)-P(2)}{Q(x)-Q(2)} = \lim_{x\to 2} \frac{\frac{P(x)-P(2)}{x-2}}{\frac{Q(x)-Q(2)}{x-2}}$$

The real magic here consists of observing that the limit of the RHS is just:
$\lim_{x\to 2} \frac{P'(x)}{Q'(x)}$

Which is just the limit of the quotient of the derivatives. Since derivatives are simpler, this limit might hopefully be easier to compute (and indeed, it is!).

The general technique outlined above is due to l'Hôpital:

Theorem 1. *Suppose that $f(a) = g(a) = 0$ and that $f(x)$ and $g(x)$ are differentiable on an open interval containing a, and that $g'(x) \neq 0$ on the open interval around a except possibility at a, then:* $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Though the theorem above is stated in the $\frac{0}{0}$ case, it actually also works in the $\frac{\infty}{\infty}$ case.

Example 1. Consider the function $f(x) = \frac{(1+x)^2}{(1+x^2)}$. Apply the general strategy to obtain a sketch of the curve.

1. Identify the function and the interval. Compute the first and second derivative.

2. Critical points: Find all points where $f'(x) = 0$. Also note any points of non-differentiability if they exist.

3. Inflection points: Find all points where $f''(x) = 0$. Also note any points of non-differentiability if they exist.

4. Test the derivative of the function between all of the the above points (Including the end points, if any.)

5. Test the function on all critical points

6. Find the regions where $f(x)$ is convex or concave.

7. Identify the absolute maximum and minimum.

8. Graph the function.

Example 2. Apply l'Hopital's rule to the limit $\lim_{x\to 0} \frac{5x-3\sin(x)}{2x}$

Example 3. Apply l'Hopital's rule to the limit $\lim_{x\to 2} \frac{\sqrt[2]{x+2}-x}{x-2}$

Example 4. Apply l'Hopital's rule to the limit $\lim_{x\to 2} \frac{e^{x+\ln(5)}-5\cos(x)}{x^2-25}$

Example 5. Use l'Hopital's rule to understand the behavior of $\frac{sin(x)}{x^2}$ as x goes to 0.

Example 6. Apply l'Hopital's rule to the limit $\lim_{x\to\pi/2} \frac{\sec(x)}{1+\tan(x)}$

Example 7. Apply l'Hopital's rule to the limit $\lim_{x\to 0} \frac{1}{\sin(x)} - \frac{1}{x}$

Example 8. Apply l'Hopital's rule to the limit $\lim_{x \to 0+} (1 + x)^{1/x}$.

Example 9. Apply l'Hopital's rule to the limit $\lim_{x \to \infty} x^{1/x}$

Example 10. Apply l'Hopital's rule to the limit $\lim_{x \to 0+} \frac{\ln(x)^2}{\ln(\sin(x))}$